# Unusual phase properties of the $Z_N$-symmetric quartet FCC model



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# Unusual phase properties of the $Z_N$-symmetric quartet FCC model

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**Abstract.** A $Z_N$-symmetric theory with four-spin interactions on a three-dimensional FCC lattice is analysed. The model possesses an unusual semi-local symmetry and is self-dual. For $N < N_c \simeq 5$ it undergoes a single, first-order transition, whereas for $N \geq N_c$ it exhibits two continuous transitions which may or may not be infinite order. The intermediate phase appearing between the ferromagnetic ordered and disordered phases for $N \geq N_c$ is a massless phase with no long-range order. The theory has a charged gas representation in which the charges describe, in a peculiar way, topological excitations of the system and are subject to novel semi-global neutrality conditions. These charges interact through a two-body potential which is anisotropic and asymptotically proportional to the square of the logarithm of the separation between the charges. The behaviour of the theory is strongly reminiscent of that of the two-dimensional $Z_N$ spin models and the four-dimensional $Z_N$ gauge theories and, quite naturally, constitutes a three-dimensional interpolating model. The possible relevance of this and related models to the Lifshitz point problem in certain helical magnetic and liquid crystal systems is briefly discussed.

## 1. Introduction

The model which is analysed in this paper, a theory of the dynamics of $Z_N$ spins with four-body interactions defined on a face-centred-cubic lattice, is of interest both because of its possible relevance to experimentally accessible systems as well as its intriguing statistical properties.

As we shall see, the model possesses three phases as a function of $\beta$, the inverse temperature, for $N \geq 5$. Numerical and analytical results presented below indicate that the intermediate phase is a 'soft' massless phase in which there is no long-range order, somewhat similar to the low-temperature phase of the ordinary two-dimensional $XY$ model. In studies of the phase structure of various three-dimensional liquid-crystal and helical-magnetic systems, it has been suggested that the helical-ferromagnetic phase boundary should be a kind of critical boundary with no long-range order. It is therefore quite plausible that the middle phase of our model may describe the boundary of some such system. Moreover, if this is the case, the high-temperature phase transition in our model, separating the disordered and intermediate phases, will then correspond to the Lifshitz point of the physical system in question. Our analysis of the model has not as yet determined whether or not this continuous transition is

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an infinite-order (Kosterlitz–Thouless) one. However, in this context, it should be noted that the model of Amit *et al* (1982), which shares many of the statistical properties of our theory, does have Kosterlitz–Thouless transitions.

Aside from the physical relevance of our model, which will be discussed further below, its fascinating and unusual statistical features mark it as a system worth studying in its own right.

First, the theory is self-dual (more precisely, its Villain version is self-dual). In this respect, it is precisely analogous to the two-dimensional clock models and the four-dimensional $Z_N$ gauge theories. This property is also shared by the two-dimensional model of three-body $Z_N$ interactions on a triangular lattice (Alcaraz and Jacobs, 1982, Alcaraz *et al* 1982). In addition, our FCC model, like all of these other $Z_N$ systems, has a single order–disorder transition for $N \leqslant 4$. When $N > 4$, the nature of the three phases is again very similar in all these theories, namely, a disordered high-temperature phase, an intermediate massless phase with no long-range order and an ordered low-temperature phase.

The spatial symmetries of our model are also quite interesting in that they are neither global nor local. As we shall show, the Hamiltonian of our theory (or, viewed as a field theory, its Lagrangian) is invariant under a rotation of all the spins lying in a single principal plane of the FCC lattice. Therefore, to make a complete 'gauge choice' in this model, one must fix the orientation of one spin on each of the planes of the lattice, which means that one has the freedom of making a gauge choice on a one-dimensional subspace of our lattice. When we consider further the similarity in the duality properties and phase structure of our theory with those of the class of $Z_N$ models mentioned above, we note that the semi-local (or semi-global) nature of the symmetry marks the model as a natural candidate for a three-dimensional theory which interpolates between these two- and four-dimensional systems†.

Another important aspect of our model is that, like other Abelian lattice systems, it can be expressed in terms of a set of variables which are related to the topological excitations of the theory. In particular, the U(1) version of our model can be written in terms of a set of point charges interacting through a long-range anisotropic logarithmic potential. (Actually, the potential is an involved function of products of logarithms of the separation. Throughout the paper, though, for convenience we shall simply use the term 'logarithmic' to describe this behaviour.) The allowed configurations of charges arise from the imposition of unusual, non-local neutrality conditions. Roughly speaking, only those configurations contribute which are neutral on certain two-dimensional surfaces in our three-dimensional lattice. As in the two-dimensional vector Potts models, one can understand the phase transitions in our model as arising from the condensation of topological excitations. In view of the above discussion concerning the likely relevance of our model to Lifshitz point problems, the logarithmic gas representation of the theory should prove quite useful.

In the next section we shall define the model and discuss in detail its symmetries and duality properties. The logarithmic gas representation is also derived in this section. In § 3 we report the results of our Monte Carlo simulations of the model and discuss the conclusions which are drawn from this analysis. In this context it should be mentioned that the model displays unusually strong finite-size effects. A detailed description of the reasons behind these effects has been given (Alcaraz *et al*

† It is quite reasonable to suppose that a non-Abelian generalisation of our theory would also interpolate between the corresponding two- and four-dimensional models with the same symmetry group. This interesting possibility deserves further study.

1982b) and here we only mention that particular care is needed when dealing with the finite system. In § 4 we summarise and discuss our results. Finally, we include an appendix in which $n$-point correlation inequalities are derived and then used in conjunction with an argument similar to one used in studies of other $Z_N$ models (Elitzur *et al* 1979, Amit *et al* 1982) to argue that the Villain version of our theory has three phases for large enough $N$. The intermediate phase appearing between ordered and disordered ferromagnetic phases has no long-range order and is massless (i.e., has an infinite correlation length).

## 2. The FCC $Z_N$-symmetric quartet model

### *2.1. Definition of the model*

Consider an FCC lattice. On each site place a $Z_N$ spin $S(\boldsymbol{n}) = \exp(2\pi i q_n/N)$, $q_n = 0, 1, \ldots, N-1$ ($\boldsymbol{n}$ labels a lattice site). Allow the spins to interact four at a time so that the Hamiltonian is given by

$$H = \tfrac{1}{2}\sum_{\boldsymbol{n},e}[1-\tfrac{1}{2}\{S(\boldsymbol{n})S[\boldsymbol{n}+\tfrac{1}{2}e(\hat{x}+\hat{y})]S^{+}[\boldsymbol{n}+\tfrac{1}{2}e(\hat{x}+\hat{z})]S^{+}[\boldsymbol{n}+\tfrac{1}{2}e(\hat{y}+\hat{z})]+\text{HC}\}] \tag{2.1}$$

where $e$ takes the values $\pm 1$ and the sum over $\boldsymbol{n}$ runs over all lattice sites.

The partition function is then

$$Z = \sum_{\{q=0\}}^{N-1} \exp(-\beta H). \tag{2.2}$$

An equivalent way of writing $H$ is

$$H = \sum_{t}(1-\cos\Delta_4 q) \tag{2.3}$$

where the sum is now over all elementary tetrahedra of the lattice consisting of one corner site in a cubic lattice and the centres of the three faces of the cube which share that site, and where

$$\Delta_4 q \equiv (2\pi/N)(q_1+q_2-q_3-q_4) \tag{2.4}$$

In equation (2.4) we choose $q_1$ and $q_2$ to be the sites of the tetrahedron which lie on the same $x$–$y$ plane, while $q_3$ and $q_4$ are displaced from these in the $z$ direction. This choice is purely conventional, of course, and any other choice defines an equivalent model.

We shall generally be fairly cavalier about boundary conditions (except in § 3) since we shall always be primarily interested in the infinite-volume limit. However, when necessary and unless otherwise stated, periodic toroidal boundary conditions may be assumed.

### *2.1. Symmetries of the model*

The Hamiltonian (2.1) has two kinds of spatial symmetries. First, it has a standard global symmetry in that it is invariant under a simultaneous $Z_N$ rotation of all spins, $S(\boldsymbol{n}) \to S(\boldsymbol{n})\exp(i(2\pi p)/N)$, $p = 0, 1, \ldots, N-1$. In addition, however, it has an unusual 'planar' symmetry which is neither global nor local (as in an ordinary gauge

theory), but is somewhere in between. The theory is invariant under a uniform rotation of all spins lying in one principal $(x, z)$ or $(y, z)$ plane of the lattice. Furthermore, it is also invariant if all spins in a principal $(x, y)$ plane undergo a staggered rotation: that is, if $S(n)$ is rotated by $\exp(i2\pi p/N)$, then its four nearest neighbours (in the same $(x, y)$ plane) are rotated by $\exp(-i2\pi p/N)$. Clearly, there is nothing intrinsically special about the $z$ direction. The asymmetry is only due to our choice of signs in (2.4), which choice was made to ensure that the theory is invariant under a global, uniform $Z_N$ rotation. Indeed, we could just as well have chosen all signs positive, in which case the global symmetry would become staggered but all three directions would be equivalent. Because of this rich symmetry structure, whenever the symmetry is not broken (spontaneously or otherwise) the simplest non-vanishing correlation functions must involve four points, as can be easily demonstrated. Contact between our model with its unusual planar symmetries and theories with more usual local or global symmetries will be discussed further in § 4.

### *2.3. Duality properties and the logarithmic gas representation*

The model defined above for $N = 2, 3$ and 4 is self-dual. For $N \geq 5$ it is essentially self-dual in that the dual theory is again a theory on an FCC lattice with four-body interactions defined on the elementary tetrahedra, just as in (2.1). For $N \geq 5$, however, higher harmonic interactions such as $\cos(k\Delta_4\phi)$ with $k$ an integer less than or equal to $N/2$ are generated. $\Delta_4\phi$ is the dual analogue of $\Delta_4 q$ given in (2.4). Because of this, though, a self-dual model which generalises (2.3) can be given through

$$H = \sum_{t} \sum_{k=1}^{\lambda} J_k[1 - \cos(k\Delta_4 q)] \tag{2.5}$$

where $\lambda$ is the integer part of $N/2$. The dual of this model will have a Hamiltonian of the same form as (2.5) in which the couplings are some functions of the $J_k$. This situation is precisely analogous to that of the $d = 2$ $Z_N$ spin systems, the $d = 4$ $Z_N$ gauge theory and certain other $Z_N$ models in various dimensions (Cardy 1980, Alcaraz and Köberle 1980, 1981, Savit 1981, Amit *et al* 1982, Alcaraz and Jacobs 1982). Also analogous to the situation encountered in those theories is the fact that the periodic Gaussian, or Villain version of our model (to be introduced below) is self-dual for all $N$. It is believed that the Villain version of our model belongs to the same universality class as (2.1).

The procedure for constructing the dual form of our theory is entirely analogous to that used for other similar models which already exist in the literature (see, for example, Savit 1980 and references therein) so we will be very brief. Moreover, for simplicity we shall explicitly deal with the Villain form of our model given in equation (2.6) below instead of the cosine model of equation (2.1). (Note, however, that the Monte Carlo analysis of the next section was done with the Hamiltonian of equation (2.1).) On the way we shall encounter a very interesting charged logarithmic gas representation of the model which is also valid for (2.1) although in a more complicated form.

The partition function for the Villain form of (2.1) can be written as

$$Z = \sum_{\{l_t=-\infty\}}^{\infty} \sum_{\{q=-\infty\}}^{\infty} \exp\left(-\frac{\beta}{2}\sum_{t}(\Delta_4 q - 2\pi l_t)^2\right) \tag{2.6}$$

where the sum in the exponent is over all elementary tetrahedra, $\Delta_4 q$ is given in (2.4) and $\{l_t\}$ is a set of integers, one member of which is associated with each tetrahedron. The sum over $\{l_t\}$ reinstates the periodicity which is lost in a simple quadratic, spin-wave approximation to (2.1). For convenience, the sum over $\{q\}$ has been extended from $-\infty$ to $+\infty$, which produces a (harmless) multiplicative infinity in (2.6).

Using the Poisson summation formula and introducing a set of integer-valued variables, $\{p\}$, one $p$ per lattice site as well as a set of Fourier conjugate variables $\{\tau_t\}$, one $\tau_t$ per tetrahedron, (2.6) can be written as

$$Z = \sum_{\{l_t\}} \sum_{\{p=-\infty\}}^{\infty} \int_{-\infty}^{\infty} Dq \exp\Big(-\frac{\beta}{2}\sum_t (\Delta_4 q - 2\pi l_t)^2 + \sum_n 2\pi i p_n q_n\Big)$$
$$= \sum_{\{l_t\}} \sum_{\{p\}} \int_{-\infty}^{\infty} Dq \int_{-\infty}^{\infty} D\tau_t \exp\Big[\sum_t \Big(-\frac{1}{2\beta}\tau_t^2 + i\tau_t(\Delta_4 q - 2\pi l_t)\Big) + \sum_n 2\pi i p_n q_n\Big] \tag{2.7}$$

where we have dropped irrelevant overall numerical factors. The second term inside the square brackets in (2.7) is integrated by parts, changing the exponent into

$$\sum_t (-\tau_t^2/2\beta - 2\pi i \tau_t l_t) + \sum_n 2\pi i (p_n - \alpha_n/N) q_n \tag{2.8}$$

where $\alpha_n$ is the linear combination of the eight $\tau$'s associated with the eight tetrahedra sharing the site $n$, namely,

$$\alpha = \tau_{+++} - \tau_{++-} + \tau_{+--} - \tau_{+-+} - \tau_{-++} + \tau_{-+-} - \tau_{---} + \tau_{--+}; \tag{2.9}$$

the three subscripts define the orientation of the tetrahedron from site $n$; e.g., $\tau_{+-+}$ corresponds to the tetrahedron which is oriented in the $+x$, $-y$, $+z$ direction from site $n$. (The right-hand side of (2.9) actually represents the action of the operator dual to $\Delta_4$ on the $\tau$'s.)

Doing the integrations over the $q_n$ and dropping overall constant factors leads to

$$Z = \sum_{\{l_t\}} \sum_{\{p\}} \int D\tau_t \exp\Big(\sum_t (-\tau_t^2/2\beta - 2\pi i \tau_t l_t)\Big) \prod_n \delta(2\pi p_n - 2\pi\alpha_n/N). \tag{2.10}$$

The constraints imposed by the Dirac $\delta$-functions in (2.10) can be satisfied by introducing a new continuous field $\phi$ at each site of the lattice and writing

$$\tau_t = \phi_1 + \phi_2 - \phi_3 - \phi_4 + NM_t = N\Delta_4\phi/2\pi + NM_t \tag{2.11}$$

where $M_t$ is an integer. The $\phi_n$ are associated with the corners of the tetrahedron $t$ and the linear combination of $\phi$'s in (2.11) is the same as that of $q$'s in (2.4). If we allow $M_t$ to run independently over all integers we will commit an overcounting error. In principle, a 'gauge' restriction should be imposed on the allowed set of values $M_t$ can take in (2.11). However, the overcounting is uniform and harmless and we shall ignore this subtlety here. For a further discussion of this point see Savit (1980) and references therein.

Up to certain possible (for the present purpose, unimportant) redefinitions of variables (2.11) is necessary and sufficient to satisfy the constraints in (2.10).

Inserting (2.11) into (2.10), replacing the integral over $\tau_t$ by an integral over $\phi$ and a sum over $M_t$ leads to

$$Z = \sum_{\{m\}} \sum_{\{M\}} \int D\phi \exp\left( \sum_t -\frac{N^2}{8\pi^2\beta} (\Delta_4 \phi - 2\pi M_t)^2 + 2\pi i \sum_n \phi_n m_n \right) \tag{2.12}$$

where we have performed an integration by parts in the exponent. In (2.12) the $m_n$ are linear combinations of the $l_t$. The functional form of $m_n(l_t)$ is identical to that of $\alpha_n(\tau_t)$ given in (2.9).

We can now use (2.12) either to demonstrate the self-duality of the theory or to construct the logarithmic-gas representation. Self-duality is obtained immediately by using the Poisson summation formula in (2.12) to perform the sum over $\{m\}$. This turns the integral over $\phi$ into a discrete sum over integers to give, up to overall field-independent factors,

$$Z = \sum_{\{M_t\}} \sum_{\{\phi\}} \exp\left( -\frac{N^2}{8\pi^2\beta} \sum_t (\Delta_4 \phi - 2\pi M_t)^2 \right) \tag{2.13}$$

which is the same as (2.6) but with $\beta$ replaced by $\tilde{\beta} \equiv N^2/4\pi^2\beta$. Note that, in (2.13), the dual FCC lattice is the same as the original one (i.e., not displaced). Also, as we mentioned at the beginning of this section, we have chosen to deal with the periodic Gaussian approximation to our theory only for simplicity. Indeed, a procedure entirely analogous to that used in deriving (2.13) can be applied to (2.3) or (2.5). When this is done, one finds that the properties of these theories under duality are exactly the same as those of the class of two- and four-dimensional $Z_N$ models mentioned previously. Therefore, all conclusions which follow from the duality properties of those models can be drawn for our theory as well. In particular, we find that for $N = 2$, 3 and 4 the theory given by (2.3) is self-dual. Therefore, if these models have a unique phase transition, this must occur at the self-dual point $\beta = \tilde{\beta}$. The numerical results of the following section indicate strongly that this is indeed the case. For $N > 4$ the model (2.3) is not self-dual, but we expect that this theory is rather similar to its Villain version, (2.6), which is self-dual. For all $N$, the self-dual point for (2.6) is at $\beta = \tilde{\beta} = N/2\pi$. In the following section we show that the theory given by (2.3) has a phase transition at a temperature which soon becomes essentially independent of $N$ as $N$ grows. If (2.3) and (2.6) describe very similar theories, one would therefore expect a second transition for the larger-$N$ models at an inverse temperature which grows like $N^2$. This is precisely what our Monte Carlo analysis shows.

To generate a logarithmic gas representation for the model we return to (2.12) and integrate over $\phi$ instead of summing over $m$. If we do that, we will generate a theory with long-range interactions between two types of charges; one associated with the $m$'s and the other with the $M$'s. This representation will be analogous to the double Coulomb gas representation for the $d = 2$ $Z_N$ spin models (Kadanoff 1978). Alternatively, one could attempt to transform (2.12) into a form analogous to the strings-plus-vortices picture of the $d = 2$ $Z_N$ spin models (Einhorn *et al* 1980). For our purposes, however, it is sufficient to examine the $N \to \infty$ limit of (2.6) and show that this U(1) theory can be written as a three-dimensional logarithmic gas. The modifications arising from finite-$N$ effects can be addressed in ways analogous to those used for the $d = 2$ vector Potts models.

In the $N \to \infty$ limit one can write (2.12) as

$$Z_{U(1)} = \sum_{\{m\}} \int D\phi \exp\left[ -\frac{1}{2\beta} \sum_{i} \left( \frac{N}{2\pi} \Delta_4 \phi \right)^2 + 2\pi i \sum_{n} \phi_n m_n \right] \tag{2.14}$$

with

$$N\Delta_4\phi/2\pi = \phi_1 + \phi_2 - \phi_3 - \phi_4.$$

In momentum space, the exponent in (2.14) takes the form

$$\int \frac{d^3k}{(2\pi)^3} \left( -\frac{1}{2\beta} |\eta(\boldsymbol{k})|^2 D(\boldsymbol{k}) + 2\pi i \eta(\boldsymbol{k}) \mu(\boldsymbol{k}) \right) \tag{2.15a}$$

where

$$D(\boldsymbol{k}) = 64[1 + \cos \tfrac{1}{2}\sqrt{2}\, k_x \cos \tfrac{1}{2}\sqrt{2}\, k_y - \cos \tfrac{1}{2}\sqrt{2}\, k_z (\cos \tfrac{1}{2}\sqrt{2}\, k_x + \cos \tfrac{1}{2}\sqrt{2}\, k_y)] \tag{2.15b}$$

$$\mu(\boldsymbol{k}) = \sum_{n} m_n \exp(-i\boldsymbol{k} \cdot \boldsymbol{n}) \qquad \phi_n = \frac{1}{\Omega} \int d^3k \exp(i\boldsymbol{k} \cdot \boldsymbol{n}) \eta(\boldsymbol{k}) \tag{2.15c, d}$$

with $\Omega = 64\pi^3$. The $\boldsymbol{k}$ integrations are over the first Brillouin zone. One can now carry out formally the Gaussian integrals in (2.14) to obtain

$$Z_{U(1)} = \tilde{Z}_0 \sum_{\{m\}} \exp\left( -\beta \int \frac{d^3k}{(2\pi)^3} |\mu(\boldsymbol{k})|^2 D^{-1}(\boldsymbol{k}) \right) \tag{2.16}$$

where $\tilde{Z}_0$ is the dual spin-wave part of (2.14) ((2.14) when all $m_n = 0$). Since $D(\boldsymbol{k})$ vanishes for a set of points within the Brillouin zone, (2.16) is not well defined as it stands. The singularities occur at the points

$$k_x = 0 \qquad k_y = \pm\sqrt{2}\pi \qquad \forall k_z \in B \tag{2.17a}$$

$$k_y = 0 \qquad k_x = \pm\sqrt{2}\pi \qquad \forall k_z \in B \tag{2.17b}$$

$$k_z = 0 \qquad k_x = 0 \qquad \forall k_y \in B \tag{2.17c}$$

$$k_z = 0 \qquad k_y = 0 \qquad \forall k_x \in B \tag{2.17d}$$

where $B$ is the interval $(-\sqrt{2}\pi, \sqrt{2}\pi)$. The lines of singularities given in (2.17*a*) and (2.17*b*) occur at the edges of the Brillouin zone and are hence shared by the neighbouring zones, whereas (2.17*c*) and (2.17*d*) are inside the Brillouin zone. This fact must be considered when evaluating the contributions of these poles to (2.16). The conditions (2.17) are simply momentum-space expressions of the three semi-global planar symmetries described earlier. Because $D(\boldsymbol{k})$ vanishes at these points, the contribution to (2.16) from a given configuration of $m$'s will vanish unless the configuration satisfies certain neutrality conditions. From (2.16) one easily sees that the statements given in (2.17) imply, respectively, the following conditions on the $m$'s:

$$\sum_{n} \exp(ik_y n_y) m_n = 0 \qquad \forall k_x \tag{2.18a}$$

$$\sum_{n} \exp(ik_x n_x) m_n = 0 \qquad \forall k_y \tag{2.18b}$$

$$\sum_{n} \exp(ik_z n_z) \exp(i\pi n_x) m_n = 0 \qquad \forall k_z \tag{2.18c}$$

$$\sum_{n} \exp(ik_z n_z) \exp(i\pi n_y) m_n = 0 \qquad \forall k_z \tag{2.18d}$$

(2.18*a*) ((2.18*b*)) requires that the configurations of $m$'s be neutral on all principal $(x, z)$ $((y, z))$ planes. (2.18*c*) implies that charges on any principal $(x, y)$ plane, when weighted with spatially alternating signs, must add up to zero. Because of the spatial structure of the FCC lattice, (2.18*c*) and (2.18*d*) are actually equivalent. (Of course, these conditions also imply global neutrality of the charges in the whole space.)

One can ensure that the constraints (2.18) are satisfied by choosing appropriate boundary conditions. In the context of the Villain version of the model such boundary conditions are easy to establish since the $m$'s can be written directly in terms of the $l_\tau$'s of (2.6). (Similar conditions can also be imposed for the cosine model (2.1), but the discussion is somewhat more involved.) One rather natural set of such boundary conditions can be constructed with the elementary tetrahedra which bound our FCC lattice. We define periodic and antiperiodic boundary conditions on the $l$'s in the usual fashion. The constraints (2.18) will then be satisfied if the $l$'s satisfy periodic boundary conditions in the $z$ direction and antiperiodic ones in both the $x$ and $y$ directions. By choosing a sequence of such lattices and taking the limit as the lattice size goes to infinity we will be able to arrive at a sensibly defined infinite system with the partition function (2.16). (Note that, *a priori*, one does not generally know whether another sequence of finite systems with different boundary conditions will lead to an infinite system with the same thermodynamics. However, experience with other related systems suggests that for the purposes of the present discussion, we may assume that the thermodynamics of the infinite system are relatively insensitive to a wide variety of boundary conditions.)

The close analogy with the situation encountered in the $d=2$ $XY$ model is apparent: there the global symmetry gives rise to an energetic constraint of overall fixed total charge in the infinite system. For simple periodic boundary conditions the total charge is zero and the boundary conditions simply reflect the homotopy statement that the total winding number is zero. (This is true for $T<T_c$. For $T>T_c$ the discussion is somewhat more complicated.) In our model the situation is similar except that the symmetries are more complicated and, hence, so are the analogous constraints on the charge distribution given by (2.18), implying a more complicated set of boundary conditions. In view of the similarity of our theory to the $d=2$ $Z_N$ models, it is noteworthy that the conditions (2.18) are defined on two-dimensional sublattices.

If conditions (2.18) are enforced, the coefficients of the divergent terms in the exponent of (2.16) will be zero and we can write

$$Z_{U(1)} = Z_0 \sum_{\{m\}}{}' \exp\left(-\beta \sum_{ij} m_i G_{ij} m_j\right) \tag{2.19}$$

where the primed sum indicates that only those configurations satisfying (2.18) are included. One should bear in mind that (2.19), as it stands, is in general ill-defined. The reason for this is that, in principle, the $\phi$ integrals in (2.14) produce formal divergences due to the semi-local symmetry of the theory. If this effect is superficially ignored we find that the divergences in $G_{ij}$ (which enforce (2.18)) lead to singularities in $Z_0$. The proper way to perform the integrations over $\phi$ in (2.14) is first to fix a gauge (e.g. to set all the $\phi$'s along some line in the $z$ direction equal to zero), thus rendering $Z_0$ well defined. The $m$'s on this subspace will not appear in (2.19) and a modified set of conditions will apply to the sum over $\{m\}$. Depending on the gauge choice, $G_{ij}$ will also have a different form. However, the contributions to $Z$ from gauge-invariant configurations of charges—those constructed out of gauge-invariant configurations on elementary tetrahedra—will not depend on the choice of gauge.

This is entirely analogous to the situation occurring in a normal gauge theory, where a choice of gauge is usually necessary to make the theory completely well defined, but the calculation of gauge-invariant quantities is generally insensitive to the precise method used to deal with the gauge redundancy.

The preceding discussion notwithstanding, it is however true that in our model it is in principle technically possible to compute $G_{ij}$ without fixing the gauge because the symmetries involved are not truly local. In this respect our theory is more like a gauge theory in which one has chosen, say, an axial gauge and has not fixed the residual gauge symmetry. In a given gauge, $G_{ij}$ can be thought of, loosely, as corresponding to a simple two-body potential; this is, however, a strictly correct interpretation only when restricted to the gauge-invariant subspace of configurations. With all of this in mind, one could proceed in principle to carry out the integrations over $\boldsymbol{k}$ in (2.16) to arrive at an expression for $G_{ij}$. But for arbitrary charge separation, the integrals involved are prohibitively difficult and one must content oneself with asymptotic expressions. After some algebra, we find

$$G(0, \boldsymbol{r}) \simeq 2 \ln r_x \ln r_y + [\exp(i\pi r_x) + \exp(i\pi r_y)] \ln r_z (\ln r_x + \ln r_y) \qquad r_i \gg 1. \tag{2.20}$$

The relevant $\beta$-dependence of $\ln Z_0$ in the Villain approximation is simply that of a usual Gaussian spin-wave.

When used in conjunction with configurations constructed from elementary quartets of charges (four charges, two positive and two negative, situated at the corners of an elementary tetrahedron) we see from (2.20) that the potential between two charges grows as the logarithm of the distance when the charges are separated along a given principal axis, but anisotropically like the square of the logarithm of the distance when they separate along any other direction. Recall, however, that all allowed configurations can be built by superposing elementary quartets of charges, and with this in mind, (2.20) gives an accurate description of the asymptotic dynamics of these elementary charges. We shall refer to the representation (2.19) again in § 4 when we discuss the nature of the phase transitions in our theory.

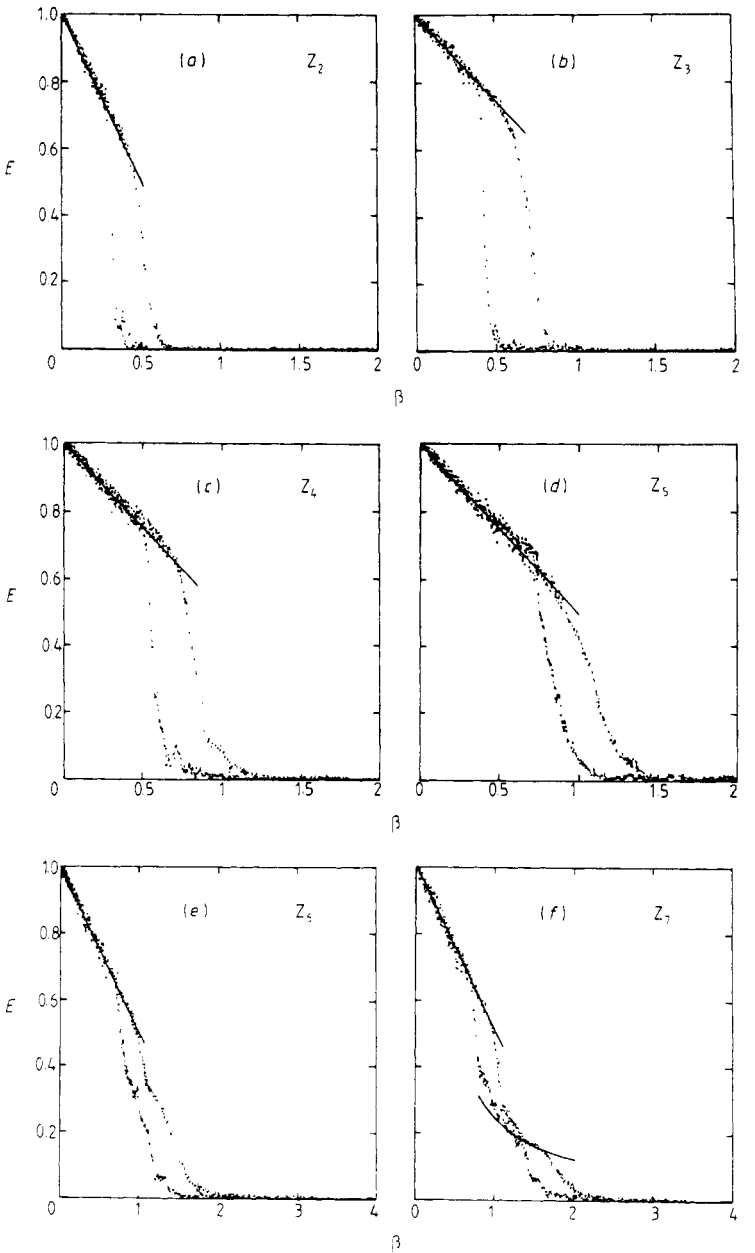
## 3. Numerical analysis of the models

In the Monte Carlo analysis of our model we have used, for the most part, the standard Metropolis updating algorithm (Metropolis *et al* 1953). In certain instances, where the procedure becomes inefficient due to the existence of a large number of degenerate configurations, a variant of the heat-bath algorithm of Creutz *et al* (1979a) was used.

The unusually large finite-size effects displayed by the model (Alcaraz *et al* 1982b) force one to be particularly careful in drawing conclusions from a study of finite lattices. We have thus used different size lattices to check our results whenever this was indicated.

For a rough, overall view of the phase structure of our theory we have performed thermal cycles in which the average energy per spin, as well as the order parameter, were measured. In such simulations an initial state with all spins set at the same value is heated by decreasing the inverse temperature from some value $\beta = \beta_0$ in small steps until $\beta = 0$ and then reversing the procedure until $\beta = \beta_0$. A single Monte Carlo iteration of the entire system is performed at each step and lattice averages of interest are recorded. If the starting temperature is sufficiently small, the initial state will be close to equilibrium, as will, in general, succeeding configurations. However, near a phase transition, the relaxation time increases and, hence, so will the difference between the measured average energy, $E$, and its equilibrium value at that temperature.

This effect will produce hysteresis-like loops in a graph of $E(\beta)$, giving a rough measure of the critical temperatures. Of course, spurious relaxation effects must be weeded out to determine the existence of a phase transition from this analysis. This can be done in any of a number of ways. We have chosen to use either a variant of the heat bath algorithm in the regions of hysteresis or an improved Metropolis algorithm. In these updating procedures a new spin is chosen or rejected by testing the entire group (heat bath algorithm) or part of it (improved Metropolis) before going to the next spin; thus ensuring that a local minimum is obtained at each update.



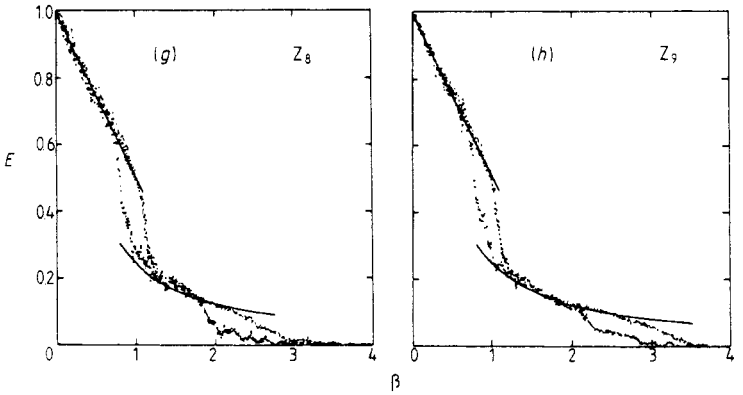



**Figure 1(*a*)–(*h*).** Thermal cycles for the $N = 2-9$ models in which the average energy (per spin) is plotted versus the inverse temperature. The full line for small $\beta$ represents the first term in the high-temperature expansion ($E = 1-\beta$ for $N = 2$, $E = 1-\beta/2$ for all others). The other full line (for the $N = 7$, 8, 9 models) is the spin-wave approximation $E = 1/4\beta$. A hint of an intermediate phase appears when $N = 6$ and becomes evident for larger $N$.

Examples of thermal cycles are shown in figure 1 for the groups $Z_2$, $Z_3$, $Z_4$, $Z_5$, $Z_6$, $Z_7$, $Z_8$ and $Z_9$. The first three cases display what seems to be a single, pronounced transition. This is confirmed by further analysis described below which shows also that these transitions are all strongly first-order. The last four examples are dramatically different. A three-phase structure is apparent in $Z_6$ and becomes more evident as the order of the group increases. The fourth figure does not show enough resolution to decide the case of the marginal $Z_5$ model. Consistent with general arguments, the high-temperature transition for $N \geqslant 5$ rapidly approaches a stationary value of $\beta \simeq 1$ whereas the low-temperature transition moves with $N$ towards increasing $\beta$. Further analysis of the $N$-dependence of this second critical point shows that it scales with the inverse gap, as it should. We find that we can fit the low-temperature phase transition with the rather natural form

$$\beta_c^{II} = \gamma/[1-\cos(2\pi/N)] \tag{3.1}$$

where the constant $\gamma \simeq 0.63$. There is little point in determining its value with precision. Thus the pattern seems to be exactly analogous to that seen in the globally symmetric two-dimensional $Z_N$ clock models (Elitzur *et al* 1979, Horn *et al* 1979, Ukawa *et al* 1980, Cardy 1980, Alcaraz and Köberle 1980, 1981, Einhorn *et al* 1980, Savit 1980), the two-dimensional semi-globally-symmetric triplet $Z_N$ model (Alcaraz and Jacobs 1982), the three-dimensional semi-locally-symmetric model of Amit *et al* (1982), as well as to the locally-symmetric, four-dimensional $Z_N$ gauge theory (Creutz *et al* 1979b).

That our model for $N \leqslant 4$ undergoes a single, first-order transition can be seen by the following argument. From the results of the previous section we know that these models are self-dual. Using a technique described below we determined the critical temperature and found it to agree (with an accuracy of $\pm 0.0025$ for $\beta$) with the self-dual points $\beta(Z_2) = \frac{1}{2}\ln(1+\sqrt{2})$ (see, however, Alcaraz *et al* 1982b), $\beta(Z_3) = \frac{2}{3}\ln(1+\sqrt{3})$ and $\beta(Z_4) = 2\beta(Z_2) = \ln(1+\sqrt{2})$, thus showing that the transition is unique. To show that it is first-order we ran long simulations at the self-dual point

with initial states which were totally ordered or totally disordered. After a very short relaxation time (of the order of 20 Metropolis iterations) each of these initial configurations reached equilibrium at two clearly distinct values of the average energy. This demonstrates the existence of a (large) latent heat. To minimise the effects of spurious metastable states we performed these simulations on systems with as few as 500 spins. As an example, the results of these simulations performed on the $Z_4$ model are shown in figure 2. In sharp contrast, figure 3 shows the case of the $Z_6$ model, run at the high-temperature transition, where the two initial configurations are seen to evolve rapidly into a single state, clear evidence of a continuous transition. In this context, it should be noted that, because of the large spatial symmetry of the theory, convergence of the standard Metropolis algorithm is extremely slow in the regions of maximum variation of the internal energy. Because of this, both the heat-bath algorithm, as well as different mixed initial configurations of the type described below, were used to check our conclusions.

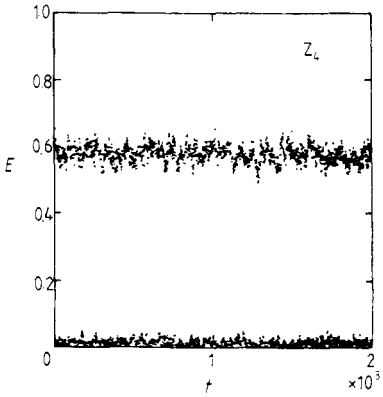



**Figure 2.** Evolution of an ordered (lower points) and a disordered (upper points) state at the self-dual point of the $Z_4$ model. The apparent stability of both states indicates a first-order transition.

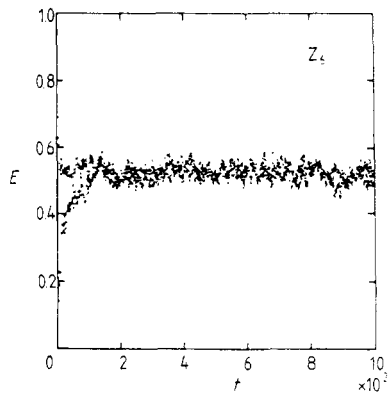



**Figure 3.** Time evolution of an ordered state and a disordered state at the high-temperature transition of the $Z_6$ model. The apparent absence of a latent heat indicates a continuous transition.

To determine the critical temperature of first-order transitions we used a technique developed in the study of lattice gauge models (Creutz *et al* 1979a, b). In this method, one studies the evolution of states in which part of the lattice is initialised to a frozen configuration and the remaining spins are set to random values. Monte Carlo simulations are then performed at fixed temperatures in the transition region (as determined, for example, from a thermal cycle). When the temperature is not the critical one, one notices a rapid initial relaxation of the system to a state with average energy between the two values, $E_+$ and $E_-$, corresponding to the energies of the two coexisting stable phases at the critical point. This is followed by a linear drift in $E(t)$ as the boundary between the two regions of the lattice shifts until the stable phase occupies the whole system. At the critical temperature, however, both phases are stable and no drift is observed. This procedure is highly efficient because the interaction between the two regions of the lattice at the boundary greatly reduces the lifetime of metastable states as the system is supercooled or superheated. As an example, the results of this kind of simulation for the $Z_3$ model are shown in figure 4.

The intermediate phase observed for $N \geq 5$ is most likely a soft phase in which correlation functions have a power-law decay at large distances and there is no long-range order. As evidence for this contention we first note that a massless, spin-wave approximation to the partition function, gives $E \simeq 1/4\beta$ for large $\beta$. The fit of $E(\beta)$ in this phase to the spin-wave approximation is excellent, indicating strongly the existence of massless spin-wave excitations. This is shown with the thermal cycles for $N = 7$, 8 and 9 in figure 1 and for the $Z_{17}$ model in figure 5.

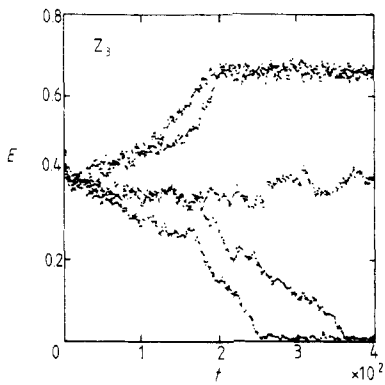



**Figure 4.** Evolution of a mixed state (discussed in the text) for temperatures above and below the self-dual point of the $Z_3$ model, $\beta = 0.67003\ldots$. Starting from the lowest set of points, the inverse temperatures are 0.65, 0.66, 0.67, 0.68 and 0.69.

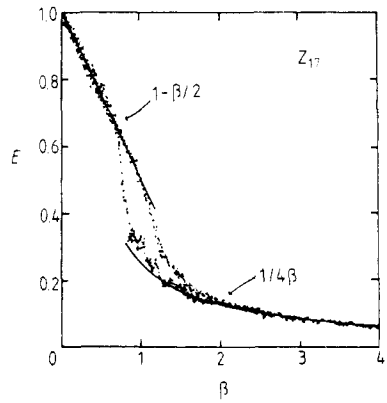



**Figure 5.** Thermal cycle for the $Z_{17}$ model. Solid lines represent the first term in the high-temperature series, $E = 1 - \beta/2$, and the spin-wave approximation $E = 1/4\beta$. The close agreement between the data and the Gaussian approximation at intermediate temperatures is evidence that this phase is massless. Note that the second, low-temperature transition occurs at a larger value of $\beta$ than is shown on this graph.

The existence of spin waves can also be seen by a feature which is peculiar to the standard local Monte Carlo procedure. Consider slowly cooling an initial state, which is completely disordered, from $\beta = 0$ through the transition at $\beta \simeq 1$. In either a disordered or a spin-wave phase the order parameter—the average spin—should vanish, because all values of the allowed angles between 0 and $2\pi$ are assumed. However, the local distribution of values for the angle in a disordered phase is very different from that expected for a spin-wave phase. Whereas in the former case the neighbours of a given spin assume, with high probability, any value in the group, in the latter they will, on the average, take values close to that of the spin being tested (modulo gauge-like fluctuations) since, in such a phase, long-wavelength fluctuations predominate. Thus, in such a simulation, the existence of spin waves should be signalled by a quench in the spin fluctuations over small temperature intervals as the temperature is lowered past the critical point. That this effect indeed occurs is seen dramatically in figure 6.

More numerical work is needed to determine with precision whether or not $N = 5$ represents the bifurcating point.

## 4. Summary and conclusions

In this section we shall recapitulate our findings and comment on the relationship of our model to other statistical models and discuss possible physical realisations.

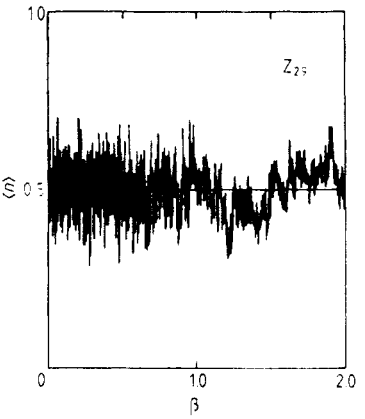



**Figure 6.** Quench in the fluctuations of the average angle as the system is cooled past the U(1) transition at $\beta \simeq 1$ (the U(1) model is well approximated at these temperatures by the $Z_{29}$ model, shown here). As discussed in the text, this effect is further evidence for the existence of spin-wave excitations in the low-temperature phase of the U(1) model.

To summarise, we have shown that the $Z_N$-symmetric quartet model defined on an FCC lattice has an unusual symmetry which is neither local nor global, but is something in between. In this respect the model naturally interpolates between the $d = 2$ globally-symmetric $Z_N$ spin models and the $d = 4$ locally-symmetric $Z_N$ gauge theories. In addition, like these models, our model is essentially self-dual. That is, the periodic Gaussian form of the model is self-dual, while the dual of the cosine form of the model is also a $Z_N$ symmetric quartet model on an FCC lattice but with somewhat more complicated interactions. However, as we have discussed, a generalised version of our model, (2.5), which contains all these higher harmonics is self-dual.

From our numerical analysis and experience with related models, we expect that there will be a low-temperature phase in the U(1) theory with algebraic decay of correlation functions. Supposing this to be the case, we show in the appendix that the periodic Gaussian version of our model must, for $N$ sufficiently large, but finite, have at least three phases: a high-temperature disordered phase, an intermediate massless phase with no long-range order, and a low-temperature ordered phase. Since we expect the cosine model to be in the same universality class, this model should also have at least three phases for large enough finite $N$, which is in agreement with our numerical results.

From (2.20) and our numerical results, one can argue that there is no low-temperature, ordered phase in the $N \to \infty$ (U(1)) model. On the other hand, the low-temperature correlations of the U(1) model do not have the exponential decay characteristic of the disordered, high-temperature states, so the $N \to \infty$ model should have one phase transition at some finite inverse temperature $\beta_0$ (this is, of course, consistent with the analysis of the past section).

The U(1) version of the model also has an interesting representation as a three-dimensional gas of charges interacting through a logarithmic potential. Because of the theory's unusual symmetries, these charges are subject to semi-local neutrality constraints along the principal planes of the lattice as expressed in (2.18). Alternatively, we may think of the allowed configurations of charges as being superpositions of elementary charge quartets as discussed in § 2. Insofar as the topological excitations are point-like, the theory is more reminiscent of the $d = 2$ $XY$ model whose topological excitations are point-like vortices, than of the $d = 4$ gauge theory whose topological

excitations are strings. However, it appears likely to us that an alternative description of our model in terms of string-like variables is possible. Indeed, this is what one would expect from simple homotopy arguments (which, however, must be gingerly applied to a theory with an unusual symmetry such as ours). A more complete understanding of our model would certainly be achieved if this question were resolved.

Our numerical results fully support the picture that our model has three phases for $N \geqslant 5$ with an intermediate massless phase, and two phases, separated by first-order transitions, for $N = 2$, 3 and 4. We have not determined yet whether or not the continuous transitions observed for the higher-$N$ models are of the Kosterlitz–Thouless (KT) type (infinite order). In this connection note, however, that whereas the $d = 2$ $XY$ model does have a KT transition, the transition, in the $d = 4$ U(1) model is second order (Lautrup and Nauenberg 1980, Bhanot 1981). It would be most desirable to perform a Monte Carlo renormalisation group analysis or to apply arguments of the type employed by Grinstein (1980) and Amit *et al* (1982) to help decide this question.

Our model is also closely related to other three-dimensional models of statistical interest. First, following the analysis by Baxter and Pierce (1981) of the $Z_2$ version, a simple extension of their arguments shows that the $Z_N$ model can be mapped onto a three-dimensional $N^4$-vertex, $N$ colour model on a simple cubic lattice. Next, our theory is closely related to the model of Amit *et al* (1981) which has a similar phase structure and a logarithmic propagator (which, however, differs in detail from (2.20)). We have also considered other $Z_N$ quartet models on different three-dimensional lattices. We have analysed a $Z_N$ quartet model on an HCP lattice (the $Z_2$ version of which has been studied by Liebmann (1982)) and found it to have the same duality properties as our model. Further, a preliminary numerical analysis indicates that it has a very similar phase structure. Finally, we note that we may define a model with four-body interactions on a BCC lattice. A little thought shows that such a theory is just a rotated, elongated version of our FCC model.

We close this section with a brief comment about possible physical realisations of the model studied in this paper. As we mentioned in the Introduction, the possible relevance of our model to the Lifshitz-point problem in various systems is quite exciting. To understand whether a given physical system can be described by a model such as ours, one should first determine that the symmetries and dimensionality of the model are appropriate for a description of the system. For the case in point, for example, it is known that the Lifshitz point in the smectic $A$-smectic $C$-nematic liquid-crystal system can be described by $d = 3$ theory with a U(1) symmetry (particularly as the Lifshitz point is approached along the $A$–$C$ phase boundary). Now, we have several $d = 3$ U(1) theories without long-range order which are *a priori* candidates for describing the $A$–$C$ phase boundary and the Lifshitz point. These models differ primarily in the precise nature of their symmetry properties and in the detailed form of their propagators. A spin-wave analysis of the physical system under consideration will indicate what form of a logarithmic potential is most appropriate for its description. It is quite likely that such an analysis would permit us to find a model of the type we have discussed here which is relevant to the physical system under consideration and would lead to a prediction of the nature of its Lifshitz point.

## Acknowledgment

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## Appendix

In this appendix we prove the following inequalities for the $m$-point, equal temperature correlation functions for the order ($\theta$) and disorder ($\phi$) variables of the Villain version of the model which will lead to an argument for the existence of three phases for large enough $N$:

$$\left\langle \cos\left( \sum_{n=1}^{m} \theta_n \right) \right\rangle_{Z_N} \geqslant \left\langle \cos\left( \sum_{n=1}^{m} \theta_n \right) \right\rangle_{U(1)} \tag{A1}$$

$$\left\langle \cos\left( \sum_{n=1}^{m} \phi_n \right) \right\rangle_{Z_N} \leqslant \left\langle \cos\left( \sum_{n=1}^{m} \phi_n \right) \right\rangle_{U(1)}. \tag{A2}$$

To apply these inequalities in the study of the phase structure of our model first note that if, as we expect, there is a massless low-temperature phase in the U(1) theory, then the above inequalities and the self-duality of the Villain version of our model (Elitzur *et al* 1979) can be used to show that for $N > 2\pi\beta_0^{\vee}$ the $Z_N$ models must have three phases. Moreover, we expect that the intermediate phase of the $Z_N$ models has qualitatively the same properties as the massless low-temperature phase of the U(1) model. In particular, the intermediate phase of the $Z_N$ models should have no long-range order. Of course, it should be clear that although (A1) and (A2) are valid whatever the distribution of the $m$-points might be, unless this set of points is constructed from $Z_N$-symmetric distributions (i.e., elementary tetrahedra) both sides of (A1) and (A2) vanish and we learn nothing from the inequalities. Since we do not have an *a priori* estimate for $\beta_0^{\vee}$ we cannot determine the critical value of $N$ from these arguments. Our numerical data favours $N_c = 5$ but further work is required to fix this value more precisely.

To prove (A1) and (A2) let us first define the $m$-point correlation function for the order variables

$$O_N(x_1, \ldots, x_m) = \left\langle \cos\left( \sum_{n=1}^{m} \theta(x_n) \right) \right\rangle \tag{A3}$$

$$\theta(x_n) = 2\pi q_n / N.$$

We find, in the periodic Gaussian approximation,

$$O_N^{\vee} = \frac{1}{Z_N^{\vee}} \sum_{\{q_n=0\}}^{N-1} \sum_{\{l_t=-\infty\}}^{\infty} \exp\left( -\frac{\beta}{2} \sum_t (\Delta_4 q - 2\pi l_t)^2 \right) \exp \frac{2\pi i}{N} \sum_n q_n Q_n \tag{A4}$$

where $Z_N^{\vee}$ is given by (2.6) and $Q_n = \sum_{i=1}^{m} \delta_{n,x_i}$. An entirely similar procedure to that used in deriving (2.10) from (2.6) leads to

$$O_N^{\vee} = \frac{1}{Z_N^{\vee}} \sum_{\{\tau_t=-\infty\}}^{\infty} \sum_{\{M=-\infty\}}^{\infty} \prod_n \delta_{J_n(M),0} \exp\left( -\frac{1}{2\beta} \sum_t \tau_t^2 \right) \tag{A5}$$

where $J_n(M) = \alpha_n + Q_n + NM_n$ and $\alpha_n$ is given in (2.9). For the U(1) limit we likewise obtain

$$O_{U(1)}^{\vee} = \frac{1}{Z_{U(1)}^{\vee}} \sum_{\{\tau_t=-\infty\}}^{\infty} \prod_n \delta_{K_n,0} \exp\left( -\frac{1}{2\beta} \sum_t \tau_t^2 \right). \tag{A6}$$

Following Elitzur *et al* (1979) we define an interpolating correlation function

$$O_H^{\vee} = \frac{1}{Z_H^{\vee}} \sum_{\{\tau=-\infty\}}^{\infty} \sum_{\{M=-\infty\}}^{\infty} \prod_n \delta_{J_n(M),0} \exp\left( -\frac{1}{2\beta} \sum_t \tau_t^2 - \frac{1}{H} \sum_n M_n^2 \right) \tag{A7}$$

where $Z_H^{\vee}$ is the numerator for $Q=0$. It is trivial to see that, as $H\to 0$, $O_H^{\vee}\to O_{U(1)}^{\vee}$, whereas as $H\to\infty$, $O_H^{\vee}\to O_N^{\vee}$. Therefore, we will prove (A1) if we can show that $\partial O_H^{\vee}/\partial H \geqslant 0$ for all $H>0$.

Differentiating (A7) with respect to $H$ we find

$$\frac{\partial}{\partial H}O_H^{\vee}=\frac{1}{(HZ_H^{\vee})^2}\sum_{\tau M}\sum_{\tau' M'}\prod_{n}\delta_{J_n,0}\delta_{I'_n,0}$$

$$\times\exp\left(-\frac{1}{2\beta}\sum_{t}(\tau_t^2+\tau_t'^2)-\frac{1}{H}\sum_{n}(M_n^2+M_n'^2)\right)\sum_{l}(M_l^2-M_l'^2) \tag{A8}$$

where $I'_n=\alpha_n+NM'_n$.

Defining at each tetrahedron the variables

$$\rho_t(\boldsymbol{n})=\tau_t(\boldsymbol{n})+\tau'_t(\boldsymbol{n})$$
$$\rho'_t(\boldsymbol{n})=\tau_t(\boldsymbol{n})-\tau'_t(\boldsymbol{n}) \tag{A9}$$

and, at each site,

$$\mu_n=M_n+M'_n \qquad \mu'_n=M_n-M'_n. \tag{A10}$$

We see that satisfying the constraints in (A8), $J_n=I'_n=0$ also implies that

$$K\equiv\rho_t^*+Q_n+N\mu_n=\rho_t^{*\prime}+Q_n+N\mu'_n\equiv K'=0$$

where $\rho_t^*=\alpha_t+\alpha'_t$, $\rho_t^{*\prime}=\alpha_t-\alpha'_t$. Notice, however, that $\rho_t$ and $\rho'_t$ ($\mu_n$ and $\mu'_n$) are not independent since by (A9) ((A10)) they should have the same parity. To sum independently over $\rho_t$ and $\rho'_t$ (as well as $\mu_n$ and $\mu'_n$), we add a factor $\frac{1}{2}[1+(-1)^{\rho_t+\rho'_t}]$ for each tetrahedron and a factor $\frac{1}{2}[1+(-1)^{\mu_n+\mu'_n}]$ for each site. This allows us to replace the sums over $\tau_t$ and $M_n$ by sums over $\rho_t$ and $\mu_n$ to get

$$\frac{\partial O_H^{\vee}}{\partial H}=\frac{1}{(HZ_H^{\vee})^2}\sum_{\rho M}\sum_{\rho' M'}\sum_{n}\delta_{K,0}\,\delta_{K',0}\exp\left(-\frac{1}{4\beta}\sum_{t}\rho_t^2-\frac{1}{2H}\sum_{n}\mu_n^2\right)$$

$$\times\sum_{l}\mu_l\mu'_l\exp\left(-\frac{1}{4\beta}\sum_{t}\rho_t'^2-\frac{1}{2H}\sum_{n}\mu_n'^2\right)$$

$$\times\prod_{t}\tfrac{1}{2}[1+(-1)^{\rho+\rho'}]\prod_{n}\tfrac{1}{2}[1+(-1)^{\mu+\mu'}]. \tag{A11}$$

The sums over site and tetrahedron variables can be divided into sums over general subsets $S$ and $T$ such that the last two factors in (A11) can be separated. This leads finally to

$$\frac{\partial O_H^{\vee}}{\partial H}=\frac{1}{(HZ_H^{\vee})^2}\sum_{l}\sum_{S}\sum_{T}\left[\sum_{\rho\mu}\left(\prod_{n}\delta_{K_n,0}\right)\mu_l\exp\left(\mathrm{i}\pi\sum_{t\in T}\rho_t+\mathrm{i}\pi\sum_{m\in S}\mu_m\right.\right.$$

$$\left.\left.\times\exp\left(-\frac{1}{4\beta}\sum_{t}\rho_t^2-\frac{1}{2H}\sum_{n}\mu_n^2\right)\right]^2\geqslant 0 \tag{A12}$$

thus proving (A1).

To prove (A2), define

$$D_N(x_1,\ldots,x_m)=\left\langle\cos\left(\sum_{n=1}^{m}\frac{2\pi}{N}\phi(x_n)\right)\right\rangle. \tag{A13}$$

In the Villain approximation one can write

$$D_N^V = \frac{1}{Z_N^V} \sum_{\{\phi_n=-\infty\}}^{\infty} \sum_{\{M_t=-\infty\}}^{\infty} \exp\left(-\frac{\tilde{\beta}}{2}\sum_t (\Delta_4\phi - 2\pi M_t)^2\right) \cos\left(\frac{2\pi}{N}\sum_l Q_l\phi_l\right)$$
$$\tilde{\beta} = \frac{N^2}{4\pi^2\beta} \tag{A14}$$

whereas

$$D_{U(1)}^V = \frac{1}{Z_{U(1)}} \sum_{\phi_n} \exp\left(-\frac{\tilde{\beta}}{2}\sum_t (\Delta_4\phi)^2\right) \cos\left(\frac{2\pi}{N}\sum_l Q_l\phi_l\right). \tag{A15}$$

Defining as before an interpolating function

$$D_H^V = \frac{1}{Z_H^V} \sum_{\phi} \sum_{M} \exp\left(-\frac{\tilde{\beta}}{2}\sum_t (\Delta_4\phi - 2\pi M)^2 - \frac{1}{H}\sum_t M^2\right) \cos\left(\frac{2\pi}{N}\sum_l Q_l\phi_l\right), \tag{A16}$$

we will prove (A2) if we can show that $\partial D_H^V/\partial H \leq 0$ for all $H > 0$.

Differentiating (A16) we obtain

$$\frac{\partial D_H^V}{\partial H} = \frac{1}{(HZ_H^V)^2} \sum_{\phi M} \sum_{\phi' M'} \exp\left(-\frac{\tilde{\beta}}{2}\sum_t [(\Delta_4\phi - 2\pi M)^2 + (\Delta_4\phi' - 2\pi M')^2] - 1/H \sum_t (M^2 + M'^2)\right)$$
$$\times \left(\sum_t (M^2 - M'^2)\right) \cos\left(\frac{2\pi}{N}\sum_l Q_l\phi_l\right). \tag{A17}$$

This expression can be symmetrised with respect to primed and unprimed variables by noting that, since (A17) is antisymmetric under $M \to M'$, the last factor can be substituted by

$$\frac{1}{2}\left(\cos\frac{2\pi}{N}\sum_l Q_l\phi_l - \cos\frac{2\pi}{N}\sum_l Q_l\phi'_l\right)$$
$$= -\sin\left(\frac{\pi}{N}\sum_l Q_l(\phi_l + \phi'_l)\right) \sin\left(\frac{\pi}{N}\sum_l Q_l(\phi_l - \phi'_l)\right) \tag{A18}$$

Inserting (A18) into (A17), defining sum and difference variables as in (A9) and (A10), correcting for the parity loss and separating the sums as before, we see that (A17) can also be written as a square. However, the overall negative sign coming from (A18) implies $\partial D_H^V/\partial H \leq 0$ and hence proves (A2).

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